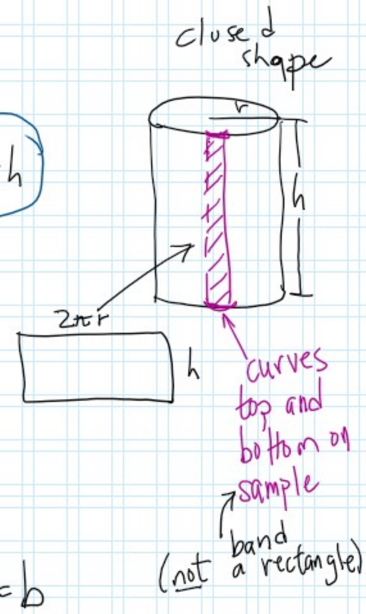
Surface Area (not in WA eBook)

recall: volume of a shape vs surface area

$V = \pi r^2 h$ vs $SA = 2\pi r^2 + 2\pi r h$

instead create shape using sample "band" and rotating it about some axis
as a result no top or bottom



$SA \sim 2\pi r \cdot h$
"slices"

to find surface area, take bands from $x=a$ to $x=b$

$$\boxed{SA = 2\pi \int_a^b \underbrace{f(x)}_{y} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx}$$

$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$ $x^{\frac{1}{2}}$

ex. the curve $y=\sqrt{4-x^2}$, $-1 \le x \le 1$ is an arc of the circle $x^2+y^2=4$ ($r=2$)
Find the surface area obtained by rotating this arc about the x-axis.

$\frac{dy}{dx} = \frac{1}{\cancel{2}\sqrt{4-x^2}} \cdot -\cancel{2}x = -\frac{x}{\sqrt{4-x^2}}$

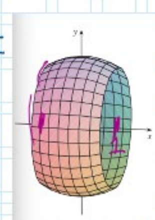
$\therefore \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{4-x^2}$

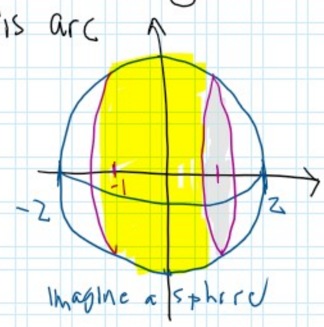
$SA = 2\pi \int_{-1}^{1} \sqrt{4-x^2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$= 2 \cdot 2\pi \int_0^1 \sqrt{4-x^2} \sqrt{\frac{4-\cancel{x^2}+\cancel{x^2}}{4-x^2}}\, dx$

$= 4\pi \int_0^1 \cancel{\sqrt{4-x^2}} \frac{2}{\cancel{\sqrt{4-x^2}}}\, dx$

$= 8\pi \int_0^1 dx = 8\pi x \Big|_0^1 = 8\pi(1-0) = \boxed{8\pi}$

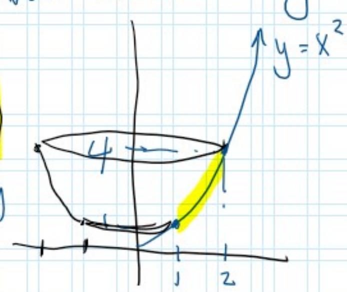




ex. the arc $y=x^2$ from $(1,1)$ to $(2,4)$ is rotated about the y-axis
Find the area of the resulting surface.

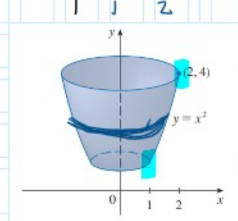
$$\boxed{SA = 2\pi \int_a^b x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy}$$

formula when integrating WRT y



$x = \sqrt{y}$

$\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$ $\therefore \left(\frac{dx}{dy}\right)^2 = \frac{1}{4y}$



$SA = 2\pi \int_1^4 \sqrt{y} \sqrt{\frac{4y}{4y} + \frac{1}{4y}}\, dy$

$= \cancel{2}\pi \int_1^4 \cancel{\sqrt{y}} \frac{\sqrt{4y+1}}{\cancel{\sqrt{4}}\cancel{\sqrt{y}}}\, dy$

$= \pi \int_1^4 \sqrt{4y+1}\, dy$

$u = 4y+1$ → $y=1$ $u_A = 5$, $y=4$ $u_B = 17$

$\frac{1}{4} du = dy$

$= \pi \cdot \frac{1}{4} \int_5^{17} u^{1/2}\, du$

$= \frac{\pi}{\cancel{4}_2} \cdot \frac{\cancel{2}}{3} u^{3/2} \Big|_5^{17} = \boxed{\frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)}$

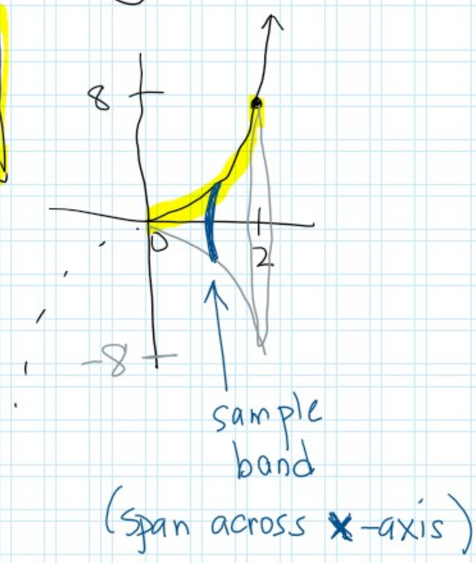
$\frac{\pi}{4} \int_{y=1}^{y=4} u^{1/2}\, du$

$\frac{\pi}{2 \cdot 3} u^3 \Big|_{u=1}^{y=4}$

$$\frac{\pi}{2\cdot 3} \; u^3 \Big|_{y=1}^{y=4}$$

ex. find the surface area obtained by rotating $y = x^3$ $\quad 0 \le x \le 2$ about x-axis

$$\boxed{SA = 2\pi \int_a^b \underbrace{f(x)}_{y} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$



$u = 1 + 9x^4$

$\frac{1}{36} du = x^3 dx$

$u_{x=2}: 1 + 9(16)$
$1 + 144$

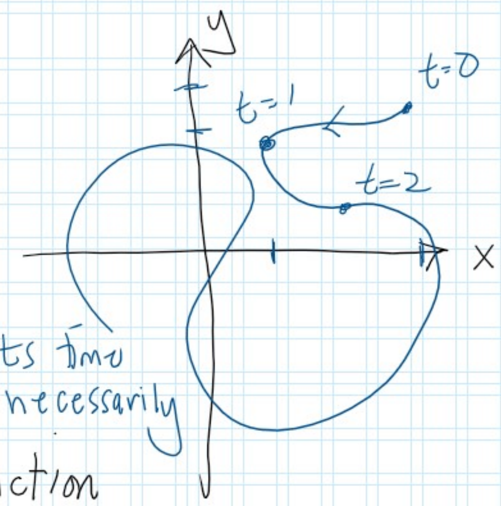
$$SA = 2\pi \int_0^2 x^3 \sqrt{1 + (3x^2)^2}\, dx$$
$$= 2\pi \int_0^2 x^3 \sqrt{\underbrace{1 + 9x^4}_{u}}\, dx$$
$$= \frac{2\pi}{36} \int_1^{145} u^{1/2}\, du$$
$$= \frac{\pi}{\cancel{18}_{9}} \cdot \frac{\cancel{2}}{3} \; u^{3/2} \Big|_1^{145}$$
$$= \boxed{\frac{\pi}{27}\left(145\sqrt{145} - 1\right)}$$

Parametric Curves $\qquad$ Section 1.6

x, y defined separately by parametric equations

$\boxed{x = f(t)} \qquad y = g(t)$ ← often represents time but not necessarily

forms parametric curve which may or may not be a function

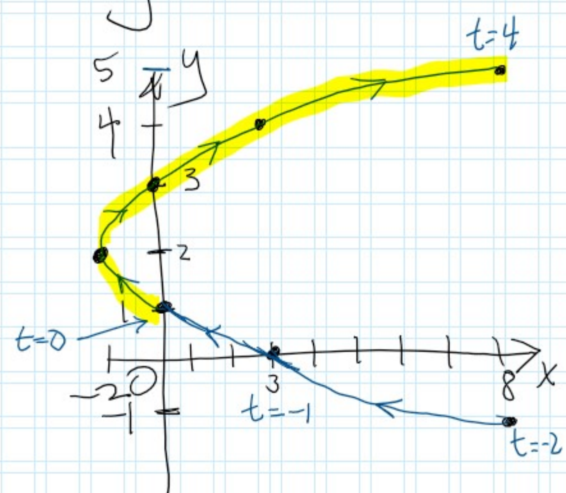


ex. sketch and identify the curved defined by the parametric equations:

$$x = t^2 - 2t \qquad y = t + 1$$

create a Table of Values:

t	x	y
-2	$(-2)^2 - 2(-2) = 4 + 4 = 8$	-1
-1	$(-1)^2 - 2(-1) = 1 + 2 = 3$	0
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5

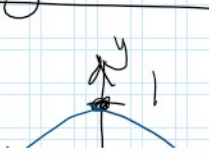


parabola!
$x = y^2 - 4y + 3$

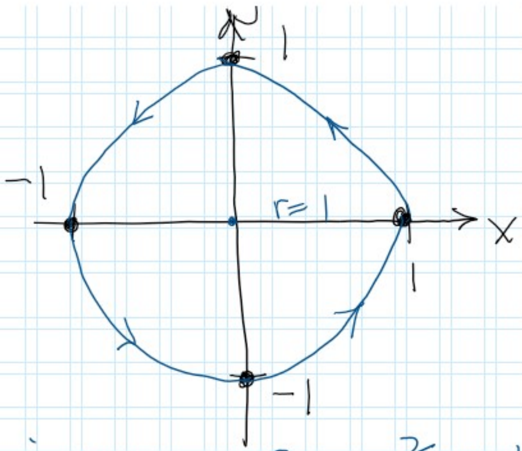
Follow up Question: what if t was restricted: $0 \le t \le 4$

ex. what curve is represented by $\boxed{x = \cos t \quad y = \sin t}$ $0 \le t \le 2\pi$
circle
angles

t	$x = \cos t$	$y = \sin t$
0	(1	0)



t	x	y
0	$(1$	$0)$
$\frac{\pi}{2}$	$(0$	$1)$
π	$(-1$	$0)$
$3\pi/2$	$(0$	$-1)$
2π	1	0

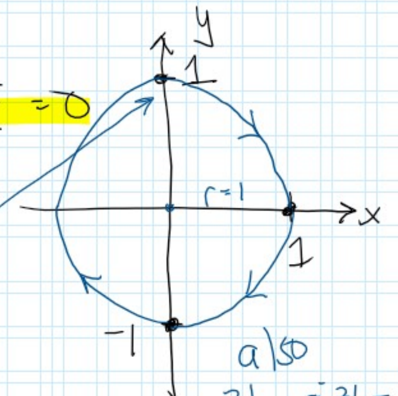


circle: $x^2 + y^2 = 1$

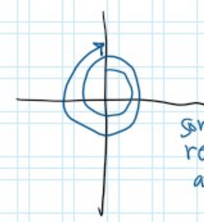
$\cos^2 t + \sin^2 t = 1$

ex. what curve is represented by $x = \sin 2t$, $y = \cos 2t$, $0 \le t \le 2\pi$

t	$x = \sin 2t$	$y = \cos 2t$
0	0	1
$\frac{\pi}{4}$	$\sin 2\cdot\frac{\pi}{4} = \sin\frac{\pi}{2} = 1$	$\cos 2\cdot\frac{\pi}{4} = \cos\frac{\pi}{2} = 0$
$\frac{\pi}{2}$	$\sin 2\cdot\frac{\pi}{2} = \sin\pi = 0$	-1
π	$\boxed{0}$	$\boxed{1}$
$\frac{3\pi}{2}$	0	-1
2π	0	1



also $\cos^2 t + \sin^2 t = 1$

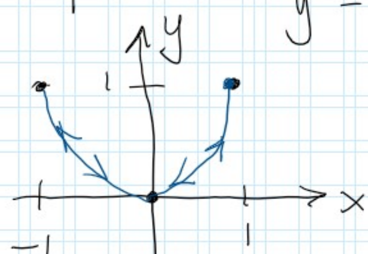


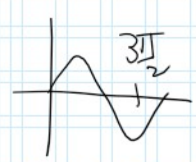
sometimes represented as

different parametric equation pairs could potentially create same parametric curve

ex. sketch the curve with parametric equations: $x = \sin t$, $y = \sin^2 t$

t	$\sin t$	$\sin^2 t$
0	0	0
$\frac{\pi}{2}$	1	1
π	0	0
$\frac{3\pi}{2}$	-1	1





graph will oscillate between ordered pairs: $(0,0)$, $(1,1)$, $(-1,1)$ to create a parabola piece of $y = x^2$ on $-1 \le x \le 1$